

Spectral representation of the heat current in a driven Josephson junction

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We discuss thermal transport through a Josephson junction in a time-dependent situation. We write the spectral representation of the heat current pumped by a generic drive. This enables separation of the dissipative and reactive contributions, of which the latter does not contribute to long-time averages. We discuss the physical interpretation, and note that the condensate heat current identified in [K. Maki and A. Griffin, Phys. Rev. Lett. **15**, 921 (1965)] is purely reactive. The results enable a convenient description of heat exchanges in a Josephson system in the presence of an external drive, with possible applications for the implementation of new cooling devices.

I. INTRODUCTION

Devices based on quantum mechanical effects could have a huge technological impact in the next decade. Quantum computers¹, sensors^{2–8}, and metrological devices^{9–15} promise to be more efficient, precise and outperform the classical ones. However, to work properly they require low and stable working temperatures. For this reason, it has become of paramount importance to be able to manipulate, store and transport energy at the nanoscale precisely and efficiently.

In this direction, the possibility to coherently control the heat flowing between two superconductors by manipulating the superconducting phase difference has attracted much attention^{16–19}. The main advantage with respect to other nanodevices is that, in some configurations, the superconducting phase can be controlled directly through an external magnetic field. This research field is still vastly unexplored but could be the ground play for a new class of quantum devices such as coherent coolers and nano-engines²⁰. Yet, to fully understand and exploit the potentialities of phase-coherent heat control we need to understand how the energy is transported when the system is subject to a time-dependent drive.

The dependence of the heat current flowing through a temperature-biased Josephson junction on the order parameter phase difference was predicted soon after the discovery of the Josephson effect,^{21,22} but measured only much later.¹⁶ Several theoretical aspects of the problem were also clarified only fairly recently.^{23–28} Most of the theoretical studies on the Josephson heat transport have concentrated on steady-state operation, with only few works addressing the microscopic description of effects from time-dependent driving.^{28,29}

The heat current through Josephson tunnel junctions was considered for arbitrary time-dependent phase difference in Ref. 28 based on a BCS tunneling Hamiltonian calculation, extending results obtained earlier for constant voltage.^{21,22,24} Some aspects of these results appear to be not fully understood, in particular the interpretation of the “condensate” or “sine” energy current.^{22,28} That this current is associated with the condensate appears clear from the structure of the tunneling calculation,

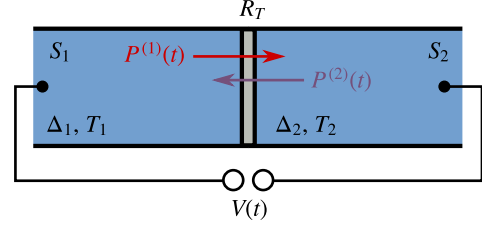


FIG. 1. Schematic setup of two superconductors S_1, S_2 with energy gaps Δ_1, Δ_2 , separated by a tunnel barrier with resistance R_T . Differences in temperatures T_1, T_2 and external bias $V(t)$ drive the heat currents $P^{(1)}(t), P^{(2)}(t)$ between the two superconducting electrodes.

tion, but its exact interpretation is less clear, given that it remains nonzero and can have either sign also at $T = 0$. Moreover, although its contribution to steady-state quantities vanishes in the cases considered, it is not immediately obvious whether it in general could contribute to time-averaged quantities in other situations.

In this work, we revisit the previous results. We write the currents in a spectral representation, and define associated causal response functions, which clarifies the general structure. From this approach, it follows that the condensate component persisting at $T = 0$ is purely *reactive*, and does not contribute to long-time averages of heat currents, for any form of drive. We discuss the analytic properties of the reactive components, and point out a “quasiparticle” part not explicitly discussed in previous works. Finally, we obtain a simple result for the heat current driven by an arbitrary periodic drive, and discuss issues relevant to practical implementation and physical interpretation of the results.

II. MODEL

We consider two superconductors S_1 and S_2 with superconducting gaps Δ_1 and Δ_2 , respectively, connected by a tunnel junction of resistance R_T . The superconducting leads are assumed to be at temperatures T_1 and T_2 (see Fig. 1). We consider the corresponding BCS

tunneling Hamiltonian model,

$$H = H_1 + H_2 + H_T, \quad (1)$$

$$H_1 = \sum_{k\sigma} [\xi_{1k} c_{1k\sigma}^\dagger c_{1k\sigma} + (\Delta_1 c_{1k\sigma}^\dagger c_{1,-k,-\sigma}^\dagger + h.c.)] \quad (2)$$

$$H_2 = \sum_{k\sigma} [\xi_{2k} c_{2k\sigma}^\dagger c_{2k\sigma} + (\Delta_2 c_{2k\sigma}^\dagger c_{2,-k,-\sigma}^\dagger + h.c.)] \quad (3)$$

$$H_T = \sum_{kq\sigma} e^{i\varphi(t)/2} M_{kq\sigma} c_{1k\sigma}^\dagger c_{2q\sigma} + h.c.. \quad (4)$$

The time-dependent phase difference $\varphi(t)$ is gauged to the tunneling Hamiltonian, so that the order parameters Δ_1, Δ_2 are real-valued. Moreover, a standard unitary transformation³⁰ has been made, shifting energies relative to the chemical potential, $\xi_k = \epsilon_k - \mu$.

Before starting, it is useful to clarify what we mean by heat current. The observable we are interested in is the variation of the energy of superconductor i in time. Following previous works^{21,24-28} we define the *heat current* exiting S_i as $P^{(i)} \equiv -\frac{d}{dt} \langle H_i \rangle$. Notice that despite the fact that this is a well defined observable, its interpretation in terms of classical thermodynamic quantities, e.g., in relation to entropy and heat, poses difficulties²⁴ and will not be discussed here.

Note that as soon as S_1 and S_2 are coupled through H_T , a fraction of energy is stored as (Josephson) coupling energy. This energy is a property of the total system, and cannot be clearly identified as belonging to either S_1 or S_2 . At the same time, the energy flowing out of, say, S_1 can either go to S_2 or increase the coupling energy. This problem is evident in the strong coupling regime, where the energy associated to H_T can dominate over the other contributions^{31,32}. If the coupling energy however is bounded and S_1 and S_2 are thermodynamically large, the long-time averages $\overline{P^{(i)}}$ of energy flows can be expected to be dominated by heat flow to the bulk of the terminals.^{33,34}

After this necessary clarification, we can discuss some general properties of the energy exchanges that occur between the superconductors. The rate of change of the total energy of the system is

$$\begin{aligned} \dot{W}(t) &= \partial_t \langle H(t) \rangle = -P^{(1)}(t) - P^{(2)}(t) + \partial_t \langle H_T(t) \rangle \\ &= \text{tr}[\rho(t) \partial_t H_T(t)], \end{aligned} \quad (5)$$

where the ensemble average is $\langle A \rangle = \text{Tr}[A\rho]$, and $\rho(t)$ is the density matrix of the total system. Above, unitarity of the time evolution, $\dot{\rho} = -i[H, \rho]$, was used. The time variation $\dot{W}(t)$ of the Hamiltonian is related to the work done on the system and the power injected in it^{33,35}. We can write $\partial_t H_T(t) = i[H_T(t), N_1] \frac{1}{2} \partial_t \varphi(t)$ and, since the electron current operator I and the voltage V are proportional to $[H_T(t), N_1]$ and $\partial_t \varphi$, respectively, we obtain the familiar form for the power injected in a electrical circuit, i.e., $\dot{W}(t) = I(t)V(t)$. The power injected into the total system can thus either increase the energies of the superconductors or change the coupling energy.

As expected, in the results below the coupling energy term does not contribute to time-averaged heat currents, and in the time average, the total absorbed heat current $-\overline{P^{(1)}} - \overline{P^{(2)}}$ is equal to the input power \overline{W} .

III. SPECTRAL REPRESENTATION

The heat current $P^{(1)}(t) = -\frac{d}{dt} \langle H_1 \rangle$ was calculated to leading order in tunneling in Ref. 28 for general time-dependent drive. The result reads

$$\begin{aligned} P^{(1)}(t) &= P_J^{(1)}(t) + P_{qp}^{(1)}(t), \\ P_{qp}^{(1)}(t) &= \frac{-i}{\pi R_T} \int_{-\infty}^t dt' e^{-\eta(t-t')} [\dot{W}_1^{qp}(t-t') W_2^{qp}(t-t') \\ &\quad + \dot{W}_1^{qp}(t'-t) W_2^{qp}(t'-t)] \cos \frac{\varphi(t) - \varphi(t')}{2} \\ P_J^{(1)}(t) &= \frac{-i}{\pi R_T} \int_{-\infty}^t dt' e^{-\eta(t-t')} [\dot{W}_1^J(t-t') W_2^J(t-t') \\ &\quad + \dot{W}_1^J(t'-t) W_2^J(t'-t)] \cos \frac{\varphi(t) + \varphi(t')}{2} \\ W_j^J(t) &= \int_{-\infty}^{\infty} dE e^{-iEt} F(E) [1 - f_j(E)], \end{aligned} \quad (6)$$

where $f_j(E) = \frac{1}{e^{E/T_j} + 1}$, $F_j(E) = -F_j(-E) = \text{Re}[\Delta_j |E^2 - |\Delta_j|^2|^{-1/2}]$, $W_j^{qp}(t) = \dot{W}_j^J(t)/(i\Delta_j)$ and $\eta \rightarrow 0^+$. Above, R_T is the tunnel junction resistance, and we set $e = k_B = \hbar = 1$.

General properties of the above result can be more clearly seen in the spectral representation. Similarly as in standard discussions of the charge current, we define³⁶

$$e^{i\varphi(t)/2} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \Phi(\omega). \quad (7)$$

It will also be convenient to consider the Fourier transform of the heat current, $P^{(1)}(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} P^{(1)}(t)$. Long-time averages can be expressed as $\overline{P} = \lim_{\tau \rightarrow \infty} [\overline{P}]_\tau$, where

$$[\overline{P}]_\tau \equiv \int_{-\infty}^{\infty} dt' \frac{z(t'/\tau)}{\tau} P(t') = \int_{-\infty}^{\infty} \frac{d\omega_0}{2\pi} \tilde{z}(\tau\omega_0)^* P(\omega_0), \quad (8)$$

where z is some real-valued window function normalized to $\int_{-\infty}^{\infty} dx z(x) = 1$ and \tilde{z} its Fourier transform — for example a Gaussian, $z(x) = e^{-x^2}/\sqrt{\pi}$, $\tilde{z}(y) = e^{-y^2/4}$.

Using the definition of Φ in Eqs. (6) and taking the Fourier transform produces:

$$\begin{aligned} P_{qp}^{(1)}(\omega_0) &= \frac{1}{4i} \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} [J_1^{qp}(\omega_1) - J_1^{qp}(\omega_1 - \omega_0)^*] \\ &\quad \times [\Phi(\omega_1) \Phi(\omega_1 - \omega_0)^* + \Phi(-\omega_1)^* \Phi(\omega_0 - \omega_1)], \end{aligned} \quad (9)$$

$$\begin{aligned} P_J^{(1)}(\omega_0) &= \frac{1}{4i} \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} [J_1^J(\omega_1) - J_1^J(\omega_1 - \omega_0)^*] \\ &\quad \times [\Phi(\omega_1) \Phi(\omega_0 - \omega_1) + \Phi(-\omega_1)^* \Phi(\omega_1 - \omega_0)^*], \end{aligned} \quad (10)$$

where the J are causal response functions³⁰ defined as

$$J_1^{J/qp}(\omega') = \frac{i}{\pi R} \int_{-\infty}^{\infty} \frac{dE}{2\pi} \frac{w^{J/qp}(E) + w^{J/qp}(-E)}{\omega' - E + i\eta}, \quad (11)$$

$$w^{qp}(E) + w^{qp}(-E) = -2\pi i \int_{-\infty}^{\infty} dE' E' N_1(E') N_2(E' - E) [f_1(E') - f_2(E' - E)] \quad (12)$$

$$w^J(E) + w^J(-E) = 2\pi i \int_{-\infty}^{\infty} dE' E' F_1(E') F_2(E' - E) [f_1(E') - f_2(E' - E)], \quad (13)$$

where $N_j(E) = N_j(-E) = \text{Re}[E(E^2 - |\Delta_j|^2)^{-1/2}]$ is the reduced density of states. The expressions corresponding to $P^{(2)}$ are obtained by exchanging the labels $1 \leftrightarrow 2$ in Eqs. (12)–(13).

The above result has a linear response theory form, as expected for computation for the change in operator expectation values in response to a perturbation. Dissipation in linear response is associated with a specific component — often the imaginary part — of the response functions. In the results here taking the definition in Eq. (11), under quite general conditions (see below), it is only the imaginary part that contributes to the long time average of the heat currents.

The imaginary (“dissipative”) parts can be written as

$$\text{Im } J_1^{qp}(\omega) = \frac{1}{R} \int_{-\infty}^{\infty} dE E N_1(E) N_2(E - \omega) \times [f_1(E) - f_2(E - \omega)] \quad (14)$$

$$\text{Im } J_1^J(\omega) = \frac{-1}{R} \int_{-\infty}^{\infty} dE E F_1(E) F_2(E - \omega) \times [f_1(E) - f_2(E - \omega)]. \quad (15)$$

The form of the result suggests they are both associated with *quasiparticle* transport. In the normal state, $\text{Im } J_1^J(\omega)|_N = 0$ and $\text{Im } J_1^{qp}(\omega)|_N = -\frac{\omega^2}{2} + \frac{\pi^2}{6}(T_1^2 - T_2^2)$.

In contrast to the imaginary part, the real (“reactive”) part of the response functions gives only nonzero frequency contributions to the heat current. The part $\text{Re } J_1^J$ corresponds to the “condensate” heat current^{21,22,24}, and is related to the “sine” heat current of Ref. 28 by $\text{Re } J_1^J(\omega) = -P_{\text{sin}}^{(1)}(\omega)$. The part $\text{Re } J_1^{qp}$ was not discussed in previous works, as it does not contribute in the constant-voltage case, but for general drive it is nonzero.

Since the response functions are causal, the reactive parts can be obtained via Kramers-Kronig relations³⁰,

$$\begin{aligned} \text{Re } J_1^{J/qp}(\omega) &= \frac{1}{\pi} \int_{-\infty}^{\infty} d\omega' \frac{\mathcal{P}}{\omega' - \omega} \text{Im } J_1^{J/qp}(\omega') \quad (16) \\ &= \frac{2\omega}{\pi} \int_0^{\infty} d\omega' \frac{\mathcal{P}}{(\omega')^2 - \omega^2} \text{Im } J_1^{J/qp}(\omega'). \end{aligned}$$

and $w^{J/qp}(E)$ are the Fourier transforms of $w^{J/qp}(t) = \dot{W}_1^{J/qp}(t) W_2^{J/qp}(t)$. The response functions have the symmetry $J^{J/qp}(\omega) = -J^{J/qp}(-\omega)^*$. By using $F_j(E) = -F_j(-E)$, we can write explicitly

where \mathcal{P} denotes the Cauchy principal value. The part $\text{Re } J_1^{qp}(\omega)$ is formally divergent, cf. 36 — the divergence is regularized by finite bandwidth/momentum dependence of tunneling. It can also be regularized by subtracting $J_1^{qp}(\omega) \mapsto J_1^{qp}(\omega) - \alpha_0 - \alpha_1\omega$ with suitable real α_j inside the integral: since

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} [\Phi(\omega_1) \Phi(\omega_1 - \omega_0)^* + \Phi(-\omega_1)^* \Phi(\omega_0 - \omega_1)] \\ = 4\pi\delta(\omega_0), \end{aligned}$$

the subtraction does not change the result. We can write

$$\begin{aligned} \text{Re } J_1^{qp}(\omega) &= \frac{2\omega}{\pi} \int_0^{\infty} d\omega' \frac{\mathcal{P}}{(\omega')^2 - \omega^2} \quad (17) \\ &\times [\text{Im } J_1^{qp}(\omega') - \text{Im } J_1^{qp}(0) + \frac{(\omega')^2}{2}]. \end{aligned}$$

The normal-state result is $\text{Re } J_1^{qp}(\omega)|_N = 0$.

Similarly as the charge current response functions,³⁶ the $J^{J/qp}(\omega)$ functions above have logarithmic singularities^{24,26} that follow from the gap edge divergences of the BCS density of states. For $\text{Im } J^{J/qp}(\omega)$, the singularities reside at $\omega = \pm|\Delta_1 - \Delta_2|$ and for $\text{Re } J^{J/qp}(\omega)$ at $\omega = \pm|\Delta_1 + \Delta_2|$. By Kramers-Kronig relations, where $\text{Im } J$ has a discontinuous jump, $\text{Re } J$ has a log-singularity, and vice versa. If the drive is not resonant, i.e., $\Phi(\omega)$ does not have δ -function or other divergences at exactly these frequencies, the resulting heat currents remain well-defined. The response functions are plotted in Fig. 2.

Finally, we can comment on the long-time averages. Based on Eqs. (8), (9), and (10), using the symmetry of $J^{J/qp}(\omega)$ and $\tilde{z}(y) = \tilde{z}(-y)^*$, we can write

$$\begin{aligned} \overline{[P_{qp}^{(1)}]_{\tau}} &= \text{Im} \int_{-\infty}^{\infty} \frac{d\omega d\omega'}{8\pi^2} \tilde{z}(\tau[\omega - \omega'])^* \\ &\times [J_1^{qp}(\omega) - J_1^{qp}(\omega')^*] \Phi(\omega) \Phi(\omega')^*, \quad (18) \end{aligned}$$

$$\begin{aligned} \overline{[P_J^{(1)}]_{\tau}} &= \text{Im} \int_{-\infty}^{\infty} \frac{d\omega d\omega'}{8\pi^2} \tilde{z}(\tau[\omega - \omega'])^* \\ &\times [J_1^J(\omega) - J_1^J(\omega')^*] \Phi(\omega) \Phi(-\omega'). \quad (19) \end{aligned}$$

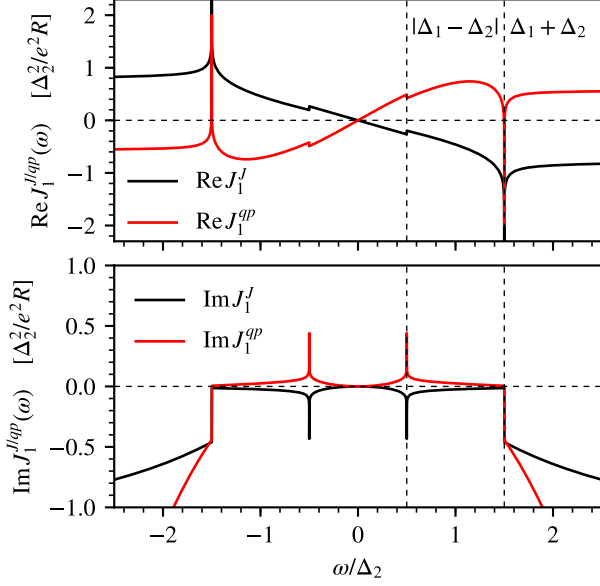


FIG. 2. Real and imaginary parts of the response functions, for $\Delta_1 = \Delta_2/2$, $T_1 = T_2 = 0.3\Delta_2$.

The average over long time scales $\tau \rightarrow \infty$ picks the zero-frequency component $\omega - \omega' \rightarrow 0$, and quite generally one can expand $\tilde{z}(\tau[\omega - \omega'])^* [J^{J/qp}(\omega) - J^{J/qp}(\omega')^*] \simeq \tilde{z}(\tau[\omega - \omega'])^* 2i \text{Im } J^{J/qp}(\omega)$ inside the integral. As a consequence, only the imaginary parts of the response functions matter for long-time averages.

A. Sum power

Consider now the sum power $P^{(T)} = -P^{(1)} - P^{(2)}$. It can be written in the same form as $P^{(1)}$ in Eqs. (9),(10) but with different response functions, $J_T^{qp/J} = -J_1^{qp/J} - J_2^{qp/J}$, which can also be written as

$$\text{Im } J_T^{qp/J}(\omega) = -\omega \text{Im } I^{qp/J}(\omega), \quad (20)$$

where^{36,37}

$$\text{Im } I^{qp}(\omega) = \frac{1}{R} \int_{-\infty}^{\infty} dE N_1(E) N_2(E - \omega) \times [f_1(E) - f_2(E - \omega)] \quad (21)$$

$$\text{Im } I^J(\omega) = -\frac{1}{R} \int_{-\infty}^{\infty} dE F_1(E) F_2(E - \omega) \times [f_1(E) - f_2(E - \omega)], \quad (22)$$

are response functions of the charge current,

$$I(t) = -\text{Im} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi} e^{-i(\omega + \omega')t} [\Phi(\omega) \Phi(-\omega')^* I^{qp}(\omega') + \Phi(\omega) \Phi(\omega') I^J(\omega')]. \quad (23)$$

Noting that

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \Phi(\omega) V(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} (-\omega) e^{-i\omega t} \Phi(\omega), \quad (24)$$

and comparison with Eqs. (8),(18),(19) results to $\overline{I(t)V(t)} = \overline{P^{(T)}}$, i.e., $\overline{P^{(1)}} + \overline{P^{(2)}} = -\overline{\dot{W}}$. There is no average heat current associated with the tunneling energy.

IV. PERIODIC DRIVE

Experiments to measure the heat current transferred in superconducting nanosystems are challenging. The physical observable is the variation of temperature of one lead. Such a measurement is usually done in the steady state regime when the transient dynamics has vanished. Under this condition it is natural to assume that the system has a periodic evolution and study what is the heat current transported in a period. Since the Josephson system dynamics is completely characterized by the superconducting phase, we consider evolution periodic in the following sense:

$$e^{i\varphi(t+\mathcal{T})/2} = e^{i\varphi(t)/2}. \quad (25)$$

In particular the constant voltage bias discussed in Refs. 24 and 28 is periodic in this sense. Then,

$$\Phi(\omega) = \sum_{k=-\infty}^{\infty} 2\pi \delta(\omega - \Omega k) \Phi_k, \quad \Phi_k = \frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} dt e^{i\Omega k t} e^{i\varphi(t)/2}, \quad \Omega = \frac{2\pi}{\mathcal{T}}. \quad (26)$$

Substituting this to Eqs. (18)–(19), we obtain

$$\overline{P^{(1)}} \equiv \overline{P_{qp}^{(1)}} + \overline{P_J^{(1)}} = \sum_{k=-\infty}^{\infty} |\Phi_k|^2 \text{Im } J_1(\Omega k, \phi_k). \quad (27)$$

where the combined response function appearing above is:

$$\text{Im } J_1(\omega, \phi) \equiv \text{Im} [J_1^{qp}(\omega) + \cos(\phi) J_1^J(\omega)]. \quad (28)$$

and the effective phase difference is

$$\cos \phi_k = \frac{2 \text{Re}[\Phi_{-k} \Phi_k]}{|\Phi_k|^2 + |\Phi_{-k}|^2}. \quad (29)$$

The long-time average coincides with the average over a single period — for $z(x) = \theta(1-x)\theta(x)$ and $\tau = \mathcal{T}$, $\tilde{z}(\tau\Omega[k - k'])^* = \delta_{k,k'}$ in (18)–(19). As above, only the imaginary (“dissipative”) part of the response function [see Eqs. (14) and (15)] contributes to the heat current, while the real (“reactive”) contribution vanishes in the periodic average.

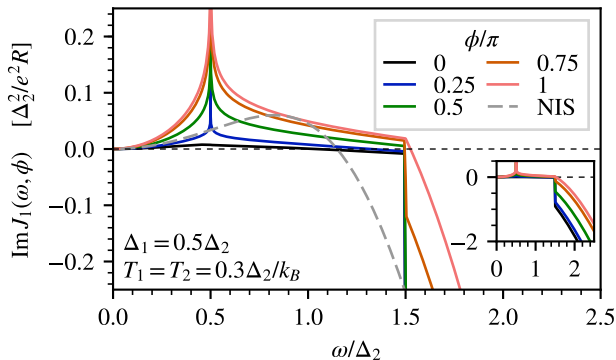


FIG. 3. Response function $\text{Im } J_1(\omega, \phi)$ for $\Delta_1 = \Delta_1/2$ and $T_1 = T_2 = 0.3\Delta_2/k_B$ for varying ϕ . The NIS case ($\Delta_1 = 0$) is also shown (dashed line). Inset: Same plot with larger y-axis range.

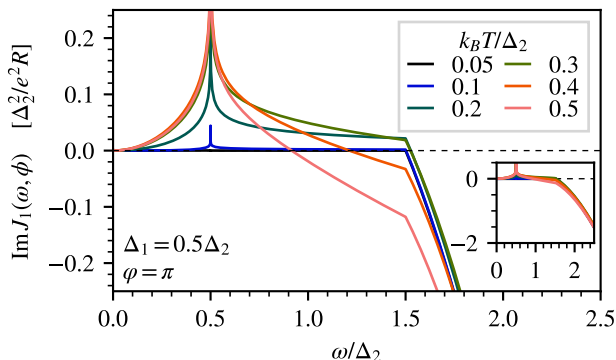


FIG. 4. Response function $\text{Im } J_1(\omega, \phi)$ for $\Delta_1 = \Delta_1/2$ and varying $T_1 = T_2 = T$ at $\phi = \pi$. Inset: Same plot with larger y-axis range.

From a physical point of view, we see that the heat current is composed by a standard quasi-particle and an "interference" contribution, similar to the ones in the steady state.^{16,24,26} Both can be interpreted²⁶ as heat transported by *quasiparticles*, as can be seen by the presence of the Fermi function in Eq. (28).

The result in Eq. (28) encompasses the ones in the previous works^{21,24,25,28}. For zero external voltage $\Omega = 0$, the result reduces to the expression for the dc heat tunneling current.^{21,23,25} For constant voltage, the result recovers that of Ref. 24 and 28, and for $\Delta_1 = 0$ the NIS junction cooling power.³⁸

The combined response function is shown in Figs. 3 and 4. At low frequencies $\omega < \Delta_1 + \Delta_2$ the function remains positive (cooling), with a logarithmic divergence appearing at $\omega = |\Delta_1 - \Delta_2|$. As a function of ϕ , the maximum is obtained at $\phi = \pi$. At high frequencies $\omega > \Delta_1 + \Delta_2$, quasiparticle transport activates and leads to a relatively larger but finite negative (heating) result $J \propto -\omega^2$ due to photoassisted pair breaking and quasiparticle

transport.

V. DISCUSSION AND CONCLUSIONS

The standard spectral representation expresses clearly the general properties of the tunneling heat current. Here, it directly indicates that the reactive "condensate" component cannot contribute to long-time averages. Moreover, a simple result is obtained relating the dc heat current to the imaginary part of a response function and the Fourier components of the drive.

The physical interpretation of the results should be viewed in the context of discussions on heat currents in coupled quantum systems.³²⁻³⁴ In particular, the problem of identifying the coupling energy stored in the junction raises questions on the status of $P^{(1)}$ defined above as experimentally relevant observables. While their long-time averages can be argued to be associated with heat that is accessible to experiments probing the bulk of the superconducting terminals, what part of the oscillating components would be accessible by measurements away from the junction region is not answered by a tunneling Hamiltonian calculation. Problems in interpretation are also illustrated by the zero-temperature behavior:³⁹ Although for the long-time averages $\overline{P^{(1)}} \leq 0$ at $T = 0$ (only heating is possible at $T = 0$), for the instantaneous currents $P^{(1)}(t) > 0$ is possible due to the reactive components that do not have a definite sign at $T = 0$.

We can also note that arguments similar to the above can have also some implications on the more general discussion on definition of heat currents in coupled quantum systems,³⁹⁻⁴¹ when the time dependence is in the coupling Hamiltonian. Based on linear response theory, time-dependent reactive components are a general feature of energy currents defined in terms of operator expectation values in such models. The Kramers-Kronig relations can then imply constraints for their time-dependent behavior and interpretation.

In summary, we wrote a spectral representation for the energy current in Josephson junctions, to obtain a clear picture of the energy currents predicted in tunnel Hamiltonian calculations. Being relatively simple, the results open the way to the practical design and optimization of SIS' coolers working on pulsed drive cycles, and for an improved understanding of their general performance properties.

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